

A SIMPLE PROOF OF BERNSTEIN-LUNTS EQUIVALENCE

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ABSTRACT. We give an easy proof of the Bernstein-Lunts equivalence of ordinary and equivariant derived categories of Harish-Chandra modules. This proof requires no boundedness assumptions. In the appendix we collect some needed, but not completely standard facts from homological algebra.

INTRODUCTION

The aim of this paper is to give an easy proof of Bernstein-Lunts equivalence, the main result of [BL2]. This easy proof has an additional advantage: it goes through without any boundedness assumptions.

To explain the setting, let \mathfrak{g} be a complex Lie algebra, and let K be a complex algebraic group acting on \mathfrak{g} via a morphism $\phi : K \rightarrow \text{Int}(\mathfrak{g})$, so that the differential of ϕ defines an embedding of \mathfrak{k} into \mathfrak{g} . Then (\mathfrak{g}, K) is called a Harish-Chandra pair. Let $\mathcal{M}(\mathfrak{g}, K)$ be the category of Harish-Chandra modules for the pair (\mathfrak{g}, K) ; these are modules simultaneously for \mathfrak{g} and K , with the usual compatibility conditions. Let $D(\mathcal{M}(\mathfrak{g}, K))$ be the derived category of the abelian category $\mathcal{M}(\mathfrak{g}, K)$. There is another related notion, the equivariant derived category $D(\mathfrak{g}, K)$ for the pair (\mathfrak{g}, K) . Objects of this category are equivariant (\mathfrak{g}, K) -complexes; these are complexes of “weak” Harish-Chandra modules (with weakened compatibility conditions), endowed with a family of homotopies i_ξ , $\xi \in \mathfrak{k}$ satisfying certain properties; for the precise definition, see [BB], [G], [BL2], [MP] or [P2]. Equivalently, equivariant (\mathfrak{g}, K) -complexes are $(\mathfrak{g}, K, N(\mathfrak{k}))$ -modules where the differential graded (DG) algebra $N(\mathfrak{k})$ is the standard complex of \mathfrak{k} . This setting is explained in §1 below.

There is an obvious functor $D(\mathcal{M}(\mathfrak{g}, K)) \rightarrow D(\mathfrak{g}, K)$: a complex of Harish-Chandra modules can be viewed as an equivariant (\mathfrak{g}, K) -complex with all $i_\xi = 0$. One of the main results of [BL2], Theorem 1.10, asserts that Q is an equivalence of bounded derived categories.

The proof in [BL2] is rather complicated; it uses K-injective resolutions and some dualizing arguments. We first give another proof for reductive K , using K-projectives instead of K-injectives. This proof is similar to the proof of an analogous result for DG modules over DG algebras in [BL1], 10.12.5.1. No boundedness needs to be assumed; see the end

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of §1 for an explanation why. For non-reductive K , the result now follows by using the arguments of [MP, §2].

In the following let us briefly describe the contents of the paper. In §1 we describe a construction of enough K-projective equivariant (\mathcal{A}, K) -complexes. In §2 we give the proof of Bernstein-Lunts equivalence (Theorem 2.4 and Corollaries 2.5 and 2.6).

In the appendix we collect some facts about homological algebra needed in the paper. These are known, but they are not readily available in the literature, at least not in the form needed.

In §A1 we explain the definition and construction of derived functors in the setting of a triangulated category and its localization, which are not necessarily the homotopic and derived categories of an abelian category. This is necessary to study equivariant derived categories. In §A2 we collect some facts about adjoint functors, in particular the ones related to homological algebra. These are used over and over, both in this paper and in [P2].

1. K-PROJECTIVES IN EQUIVARIANT DERIVED CATEGORIES

Let us start by recalling the definition of $(\mathcal{A}, K, \mathcal{D})$ -modules from [P2]. Let K be a complex algebraic group with Lie algebra \mathfrak{k} . Let \mathcal{A} be an associative algebra over \mathbb{C} , with an algebraic action ϕ of K , and a K -equivariant Lie algebra morphism $\psi : \mathfrak{k} \rightarrow \mathcal{A}$, such that the differential of ϕ satisfies

$$d\phi(\xi)(a) = [\psi(\xi), a], \quad \xi \in \mathfrak{k}, a \in \mathcal{A}.$$

Let \mathcal{D} be a DG algebra over \mathbb{C} with an algebraic action χ of K and a morphism $\rho : \mathfrak{k} \rightarrow \mathcal{D}$ of DG Lie algebras, satisfying analogous conditions.

An $(\mathcal{A}, K, \mathcal{D})$ -module is a complex V of vector spaces, with an action π of \mathcal{A} by chain maps, an algebraic action ν of K by chain maps, and a DG action ω of \mathcal{D} , such that π and ω commute and are both K -equivariant, and such that $\pi + \omega = \nu$ on \mathfrak{k} .

We denote the abelian category of $(\mathcal{A}, K, \mathcal{D})$ -modules by $\mathcal{M}(\mathcal{A}, K, \mathcal{D})$; the morphisms are chain maps which preserve all the actions. One defines the homotopic category $K(\mathcal{A}, K, \mathcal{D})$ as in [P2], Section 2.3; it is a triangulated category. Localizing with respect to quasiisomorphisms, we get to the equivariant derived category $D(\mathcal{A}, K, \mathcal{D})$. This definition is due to Beilinson and Ginzburg; an analogous definition in geometric setting is due to Bernstein and Lunts. See [BB], [G], [BL1] and [BL2].

The main example is when $\mathcal{D} = N(\mathfrak{k})$, the standard complex of \mathfrak{k} .

Now we want to show that in case K is reductive, the category $K(\mathcal{A}, K, \mathcal{D})$ has enough K-projectives (see Section A1.4). The proof is analogous to the proof of the same fact for the category of DG modules over a DG algebra \mathcal{D} from [BL1], 10.12.2. The main idea is familiar: one uses the fact that there are enough projectives in the category of weak (\mathcal{A}, K) -modules, hence there are enough K-projectives in the homotopic category of complexes $K(\mathcal{M}(\mathcal{A}, K)_w) = K(\mathcal{A}, K, \mathcal{U}(\mathfrak{k}))$ (see A1.4.7). Now one constructs K-projectives in $K(\mathcal{A}, K, \mathcal{D})$ applying the change of DG algebras from [P2], Section 2.5, that is, the functor $V \longmapsto V \otimes_{\mathcal{U}(\mathfrak{k})} \mathcal{D}$.

The construction goes as follows. Let V be an $(\mathcal{A}, K, \mathcal{D})$ -module. Forgetting the \mathcal{D} -action, we get a complex of weak (\mathcal{A}, K) -modules. Let $Q \xrightarrow{s} V$ be a K -projective resolution of V in the category $C(\mathcal{M}(\mathcal{A}, K)_w)$. We can assume that s is surjective. Let $P_0 = Q \otimes_{\mathcal{U}(k)} \mathcal{D}$. Then $s \otimes 1 : P_0 \rightarrow V \otimes_{\mathcal{U}(k)} \mathcal{D}$ is still surjective since the functor $-\otimes_{\mathcal{U}(k)} \mathcal{D}$ is right exact being a left adjoint. Furthermore, the adjunction morphism $\Psi_V : V \otimes_{\mathcal{U}(k)} \mathcal{D} \rightarrow V$, which is given by

$$\Psi_V(v \otimes x) = (-1)^{\deg v \deg x} \omega_V({}^\iota x)v,$$

is clearly also surjective. So the composition $\varepsilon_0 = \Psi_V \circ (s \otimes 1) : P_0 \rightarrow V$ is also surjective (in the category $\mathcal{M}(\mathcal{A}, K, \mathcal{D})$). We claim that ε_0 is also surjective on the level of cohomology. Since $s : Q \rightarrow V$ is a quasiisomorphism, each element of $H(V)$ has a representative of the form $s(q)$ for some cycle $q \in Q$. However, $q \otimes 1$ is a cycle in $Q \otimes_{\mathcal{U}(k)} \mathcal{D}$ since $d_{\mathcal{D}}(1) = 0$, and

$$\varepsilon_0(q \otimes 1) = \psi_V(s(q) \otimes 1) = s(q),$$

so the cohomology class of $s(q)$ is in the image of $H(\varepsilon_0)$.

Let K_0 be the kernel of ε_0 . Then from the long exact sequence of cohomology corresponding to the short exact sequence $0 \rightarrow K_0 \rightarrow P_0 \rightarrow V \rightarrow 0$ we see that $H(K)$ is the kernel of $H(\varepsilon_0)$.

We now repeat the above discussion for K_0 instead of V and proceed inductively. In this way we get a resolution

$$\dots \xrightarrow{\varepsilon_{-2}} P_{-1} \xrightarrow{\varepsilon_{-1}} P_0 \xrightarrow{\varepsilon_0} V$$

of V by equivariant complexes, which induces a resolution of cohomology of V .

Let us now consider the complex

$$(\dagger) \quad \dots \xrightarrow{\varepsilon_{-2}} P_{-1} \xrightarrow{\varepsilon_{-1}} P_0 \rightarrow 0 \dots$$

of equivariant complexes. We want to consider the total complex of P^\cdot . The construction is as follows.

Let

$$V^\cdot = \dots \rightarrow V^{-1} \xrightarrow{\delta_{-1}} V^0 \xrightarrow{\delta_0} V^1 \rightarrow \dots$$

be a complex of $(\mathcal{A}, K, \mathcal{D})$ -modules; in particular, we can view it as a double complex of weak (\mathcal{A}, K) -modules. We define the total complex $s(V^\cdot)$ in the following way. As a graded $(\mathcal{A}, K, \mathcal{D})$ -module, it is the direct sum

$$s(V^\cdot) = \bigoplus_{i=-\infty}^{\infty} V^i[-i];$$

so in particular, $s(V^\cdot)^k = \bigoplus_{i=-\infty}^{\infty} (V^i)^{k-i}$. The differential d is given by

$$d^k|_{(V^i)^{k-i}} = \delta_i \oplus d_{V^i[-i]}^{k-i} = (-1)^i \delta_i \oplus d_{V^i}^{k-i}.$$

This is one of the standard ways to define a differential on the total complex of a double complex. Therefore, $s(V^\cdot)$ as defined above is a complex and a graded $(\mathcal{A}, K, \mathcal{D})$ -module. It is now easy to see that the DG property is satisfied, so that $s(V^\cdot)$ is actually an $(\mathcal{A}, K, \mathcal{D})$ -module.

In particular, returning to our complex P^\cdot , we see that its total complex $P = s(P^\cdot)$ is an $(\mathcal{A}, K, \mathcal{D})$ -module. Clearly, ε_0 induces a morphism $\varepsilon : P \rightarrow V$. We want to see that ε is a quasiisomorphism. It is enough to check this on the level of complexes of vector spaces. Then P is the total complex of the double complex (\dagger) . Since (\dagger) is a left half-plane double complex, the cohomology of P can be computed from the second spectral sequence of (\dagger) , that is the one starting with columns. The E_1 term of this spectral sequence is

$$\dots \rightarrow H(P_{-2}) \rightarrow H(P_{-1}) \rightarrow H(P_0) \rightarrow 0 \dots$$

This is exact except at degree 0, and there cohomology is isomorphic to $H(V)$. So the spectral sequence degenerates at E_2 and the cohomology of P is isomorphic to $H(V)$. It is clear from this discussion that the isomorphism is induced by ε . So indeed ε is a quasiisomorphism.

Finally, we want to show that P is a K-projective $(\mathcal{A}, K, \mathcal{D})$ -module. In other words, we need to prove

1.1. Lemma. *Let V^\cdot be a complex of $(\mathcal{A}, K, \mathcal{D})$ -modules bounded from above, such that each V^i is K-projective. Then $s(V^\cdot)$ is K-projective.*

Proof. For any complex V^\cdot of $(\mathcal{A}, K, \mathcal{D})$ modules, it follows from the definition of the differential of $s(V^\cdot)$ that for any $p \in \mathbb{Z}$,

$$F_p s(V^\cdot) = \bigoplus_{i=p}^{\infty} V^i[-i]$$

is an $(\mathcal{A}, K, \mathcal{D})$ -submodule of $s(V^\cdot)$. It is now clear that in this way we get a decreasing exhaustive Hausdorff filtration of $s(V^\cdot)$.

Suppose now V^\cdot is bounded above by degree m , and each V^i is K-projective. Then $F_p s(V^\cdot) = 0$ for $p > m$, and by defining $\tilde{F}_p s(V^\cdot) = F_{m+1-p} s(V^\cdot)$ we get into the situation of 1.3 below, so $s(V^\cdot)$ is K-projective. Namely, the graded pieces corresponding to the above filtration are translates of V^i 's. \square

To finish the proof of existence of enough K-projectives, it remains to prove that (under certain conditions) a filtered $(\mathcal{A}, K, \mathcal{D})$ -module such that the corresponding graded pieces are K-projective, is itself K-projective. For finite filtrations this is proved in [P2], 2.6.4; the above filtration is however infinite.

To get the desired result, we need a technical result about cones. It is implicit in [BL1], 10.12.2.6.

Let $f : V \rightarrow W$ be a morphism in $\mathcal{M}(\mathcal{A}, K, \mathcal{D})$. Let $\phi : C_f \rightarrow Z$ be another such morphism. Using the decomposition $C_f = T(V) \oplus W$, we can write ϕ as $(\phi_1 \quad \phi_2)$, where $\phi_1 : T(V) \rightarrow Z$ is a graded $(\mathcal{A}, K, \mathcal{D})$ -morphism of degree 0, while $\phi_2 : W \rightarrow Z$ is a morphism in $\mathcal{M}(\mathcal{A}, K, \mathcal{D})$, i.e., also a chain map.

Writing out $d_Z\phi = \phi d_f$ as matrices, we get

$$(*) \quad d_Z\phi_1 = \phi_1 d_{T(V)} + \phi_2 T(f);$$

the other matrix entry just shows that ϕ_2 is a chain map.

1.2. Lemma. *Assume that ϕ_2 is homotopic to 0 via a homotopy h_2 , i.e., $h_2 : W \rightarrow Z$ is a graded morphism of degree -1 such that*

$$(**) \quad \phi_2 = h_2 d_W + d_Z h_2.$$

Let us denote the graded morphism from $T(W)$ to Z of degree 0 defined by h_2 again by h_2 . Then:

- (i) $\phi_1 - h_2 T(f) : T(V) \rightarrow Z$ is a morphism in $\mathcal{M}(\mathcal{A}, K, \mathcal{D})$;
- (ii) If $\phi_1 - h_2 T(f)$ is homotopic to 0, then h_2 extends to a homotopy h from ϕ to 0.

Proof. (i) Using $(*)$ and $(**)$, we see

$$\begin{aligned} d_Z(\phi_1 - h_2 T(f)) &= d_Z\phi_1 - d_Z h_2 T(f) = \\ \phi_1 d_{T(V)} + \phi_2 T(f) - (\phi_2 T(f) - h_2 d_W T(f)) &= \phi_1 d_{T(V)} - h_2 T(f) d_{T(V)} = \\ (\phi_1 - h_2 T(f))d_{T(V)}. \end{aligned}$$

(ii) Let $h_1 : T(V) \rightarrow Z$ be a graded morphism of degree -1 such that

$$(***) \quad \phi_1 - h_2 T(f) = h_1 d_{T(V)} + d_Z h_1,$$

i.e., h_1 is a homotopy from $\phi_1 - h_2 T(f)$ to 0. Let $h = (h_1 \ h_2) : C_f \rightarrow Z$. It is clearly a graded morphism of degree -1 . Using $(**)$ and $(***)$, we see that

$$\phi = (\phi_1 \ \phi_2) = (h_1 \ h_2) \begin{pmatrix} d_{T(V)} & 0 \\ T(f) & d_W \end{pmatrix} + d_Z (h_1 \ h_2),$$

which shows that $h = (h_1 \ h_2)$ is a homotopy from ϕ to 0. \square

This has the following consequence, which is exactly the property of K-projectives we need to finish the proof of 1.1.

1.3. Theorem. *Let V be an $(\mathcal{A}, K, \mathcal{D})$ -module. Let*

$$0 = F_0 V \subset F_1 V \subset F_2 V \subset \dots$$

be an increasing exhaustive filtration of V by $(\mathcal{A}, K, \mathcal{D})$ -submodules (exhaustive means that $V = \cup_i F_i V$). Assume that the $(\mathcal{A}, K, \mathcal{D})$ -modules

$$Gr_i V = F_i V / F_{i-1} V, \quad i = 1, 2, \dots$$

are K -projective, and that as graded modules, $F_i V \cong F_{i-1} V \oplus Gr_i V$ for any i . Then V is K -projective.

Proof. Let Z be an acyclic $(\mathcal{A}, K, \mathcal{D})$ -module, and $f : V \rightarrow Z$ a morphism in the category $\mathcal{M}(\mathcal{A}, K, \mathcal{D})$. We have to prove that f is homotopic to 0.

Clearly, V is the direct limit of the direct system $(F_i V)$ in $\mathcal{M}(\mathcal{A}, K, \mathcal{D})$, and f is the direct limit of the morphisms $f_i = f|_{F_i V}$. The same is true in the category $\mathcal{M}^{GR}(\mathcal{A}, K, \mathcal{D})$; there actually $V = \bigoplus_{k=1}^{\infty} Gr_k V$ and $F_i V = \bigoplus_{k=1}^i Gr_k V$ for $i > 0$. Therefore, it is enough to construct homotopies $h_i : F_i V \rightarrow Z$ from f_i to 0 for every i , which are compatible, i.e., $h_i|_{F_{i-1} V} = h_{i-1}$. Then they define a graded morphism $h : V \rightarrow Z$ of degree -1, and h is a homotopy from f to 0 because

$$f = d_Z h + h d_V$$

follows from the fact that for every i

$$f_i = d_Z h_i + h_i d_{F_i V}$$

(and $d_{F_i V} = d_V|_{F_i V}$). So we only need to construct h_i 's. This is done by induction, using Lemma 1.2. Since $F_0 V = 0$, $f_0 = 0$ and we can take $h_0 = 0$. Now 1.2. guarantees that having h_i we can extend it to h_{i+1} . Namely, we first apply [P2], 2.6.2, to the semi split short exact sequence

$$0 \rightarrow F_i V \rightarrow F_{i+1} V \rightarrow Gr_{i+1} V \rightarrow 0,$$

so we can identify $F_{i+1} V$ with the cone over a morphism from $Gr_{i+1} V$ into $F_i V$. Now notice that the condition of 1.2.(ii) is met since $Gr_{i+1} V$ is K -projective and hence any morphism from $Gr_{i+1} V$ into Z is homotopic to 0. So we can extend h_i to a homotopy from f_{i+1} to 0 and this extension is h_{i+1} . \square

As explained above, this finishes the proof of the following result.

1.4. Theorem. *Assume that K is reductive. Then any $(\mathcal{A}, K, \mathcal{D})$ -module V has a K -projective resolution $P \rightarrow V$ in $K(\mathcal{A}, K, \mathcal{D})$.* \square

If we in addition assume that \mathcal{D} is nonpositively graded and that V is bounded above by degree k , then in the above construction each P_{-i} can be taken bounded above by degree k . Namely, we can take Q to be the classical projective resolution of V , i.e., Q is bounded above by k with all Q^i projective weak (\mathcal{A}, K) -modules (see A1.4.6). Then P_0 is again bounded above by k since \mathcal{D} is nonpositively graded, etc. Now if all P_{-i} are bounded above by degree k , then so is P . In other words,

1.5. Corollary. *Assume \mathcal{D} is nonpositively graded. Then any bounded above $(\mathcal{A}, K, \mathcal{D})$ -module V has a K -projective resolution bounded above by the same degree as V .*

One can apply a dual construction to the one explained above to get existence of enough K -injectives. Since this was done in [BL2] (in essentially the same way), we will not present this construction here. Let us just comment on the difference between the two constructions. The dual analogue of 1.2 is proved in the same way. For the dual analogue of 1.3 we however need an additional finiteness assumption. Namely, the following is true:

1.3'. Prop. *Let V be an $(\mathcal{A}, K, \mathcal{D})$ -module with a decreasing Hausdorff filtration*

$$V = F_0 V \supset F_1 V \supset F_2 V \supset \dots$$

(Hausdorff means that $\cap_i F_i V = 0$). Assume that the graded objects $Gr_i V = F_i V / F_{i+1} V$ are K-injective for all i , that for any i , $F_i V$ is isomorphic to $F_{i+1} V \oplus Gr_i V$ as a graded $(\mathcal{A}, K, \mathcal{D})$ -module, and that for any $k \in \mathbb{Z}$, there are only finitely many i 's such that $Gr_i^k V \neq 0$. Then V is K-injective.

The reason for this weakening of the result is the following. In 1.3, V was the direct limit of $F_i V$. Here, we want V to be the inverse limit of $V/F_i V$, however this is not true in general. More precisely, in the graded category, the inverse limit of $V/F_i V$ is the algebraic part of the direct product $\prod_{k=0}^{\infty} Gr_k V$, while V is the direct sum $\bigoplus_{k=0}^{\infty} Gr_k V$. These two are not equal in general, but are of course equal if our finiteness assumption holds.

This difference now shows up in 1.1: to prove the dual claim we need a finiteness assumption. The result is: if a complex V^\cdot of $(\mathcal{A}, K, \mathcal{D})$ -modules is bounded from below, if each V^i is K-injective, and if for any $k \in \mathbb{Z}$ the number of i 's such that $(V^i)^{k-i} \neq 0$ is finite, then $s(V^\cdot)$ is K-injective. Therefore we do not get an analogue of 1.4, but rather of 1.5. Namely, we have to assume from the start that \mathcal{D} is non-positively graded, and that V is bounded from below. Then V embeds quasiisomorphically into a K-injective $(\mathcal{A}, K, \mathcal{D})$ -module I , bounded from below by the same degree as V .

2. BERNSTEIN-LUNTS EQUIVALENCE

One of the main results of [BL2], Theorem 1.10, implies an equivalence of bounded equivariant derived category $D^b(\mathcal{A}, K, N(\mathfrak{k}))$ and the ordinary bounded derived category of Harish-Chandra modules $D^b(\mathcal{M}(\mathcal{A}, K))$, in case \mathcal{A} is a projective $\mathcal{U}(\mathfrak{k})$ -module for the left multiplication composed with the map ψ . Their proof is rather complicated, since it uses K-injectives which are complicated by themselves, and also in the proof they often need to dualize various arguments. On the other hand, in [BL1] there is an analogous result 10.12.5.1 for DG modules, which is rather simple, uses K-projectives, and does not require boundedness. Of course, to have K-projectives the group K has to be reductive. But it turns out that once the result is proved for reductive groups, it is easy to extend it to non-reductive case using the results of [MP], Section 2.

So let us first assume that K is reductive. Then we know from Section 1 that there are enough K-projective $(\mathcal{A}, K, \mathcal{D})$ -modules. In our proof, it will be important to know that the forgetful functor from the category of $(\mathcal{A}, K, \mathcal{D})$ -modules into the category of DG modules over \mathcal{D} preserves K-projectives. We start by establishing this fact. The method to prove this is familiar: we show that this functor has a right adjoint, which is ‘acyclic’, i.e. maps acyclic complexes into acyclic complexes (see Section A1.2).

We produce this functor in two steps. The first step is analogous to the functor Ind_w from [P2], §1.3 and [MP], §2. Let V be a DG module over \mathcal{D} with action ω_V and differential d_V . We set $\text{Ind}_w(V) = R(K) \otimes V = R(K, V)$, where $R(K)$ is the algebra of regular functions on K , and $R(K, V)$ is the space of regular functions from K into V . K acts on

$\text{Ind}_w(V)$ by the right regular action, while $x \in \mathcal{D}$ acts by

$$(\omega(x)F)(k) = \omega_V(\chi(k)x)(F(k)), \quad k \in K.$$

The differential of $\text{Ind}_w(V)$ is $1 \otimes d_V$. It is readily checked that in this way $\text{Ind}_w(V)$ becomes a ‘weak (K, \mathcal{D}) -module’, i.e., an algebraic K -module with a K -equivariant DG action of \mathcal{D} . Such modules can be identified with $(\mathcal{U}(\mathfrak{k}), K, \mathcal{D})$ -modules, if we let $\mathfrak{k} \subset \mathcal{U}(\mathfrak{k})$ act by the difference of the two actions coming from K and \mathcal{D} .

2.1. Lemma. *The above described functor $\text{Ind}_w : \mathcal{M}(\mathcal{D}) \rightarrow \mathcal{M}(\mathcal{U}(\mathfrak{k}), K, \mathcal{D})$ is right adjoint to the forgetful functor.*

The proof of this is completely analogous to the proof that $\text{Ind}_w : \mathcal{M}(\mathcal{A}) \rightarrow \mathcal{M}(\mathcal{A}, K)_w$ is right adjoint to the forgetful functor (see [MP], 2.2).

The second step consists of changing the algebra $\mathcal{U}(\mathfrak{k})$ to \mathcal{A} ; here the map $\mathcal{U}(\mathfrak{k}) \rightarrow \mathcal{A}$ is given by the structural map $\psi : \mathfrak{k} \rightarrow \mathcal{A}$. This is analogous to the functor $\text{pro}_{\mathcal{A}, \mathcal{B}}$ mentioned in [P2], §1.2. If $\gamma : \mathcal{B} \rightarrow \mathcal{A}$ is a K -equivariant morphism of algebras, we have a forgetful functor from $\mathcal{M}(\mathcal{A}, K, \mathcal{D})$ into $\mathcal{M}(\mathcal{B}, K, \mathcal{D})$ given by the restriction of scalars. Its right adjoint is given by

$$V \longmapsto \text{pro}_{\mathcal{A}, \mathcal{B}}(V) = \text{Hom}_{\mathcal{B}}(\mathcal{A}, V)^{\text{alg}}.$$

Here the \mathcal{B} -homomorphisms are with respect to the left multiplication on \mathcal{A} (composed with γ). \mathcal{A} acts on $\text{Hom}_{\mathcal{B}}(\mathcal{A}, V)$ by the action π which is right translation of the argument. K acts by conjugation:

$$\nu(k)f = \nu_V(k) \circ f \circ \phi_{\mathcal{A}}(k)^{-1}, \quad k \in K.$$

$x \in \mathcal{D}$ acts by $\omega(x)f = \omega_V(x) \circ f$. The differential is given by $df = d_V \circ f$. Here, as usual, ν_V , ω_V and d_V are respectively the K -action, the \mathcal{D} -action, and the differential on V . Finally, the algebraic part of $\text{Hom}_{\mathcal{B}}(\mathcal{A}, V)$ is taken with respect to the K -action; this is clearly invariant under all the above actions and the differential. It is easy to check that $\text{pro}_{\mathcal{A}, \mathcal{B}}(V)$ is an $(\mathcal{A}, K, \mathcal{D})$ -module, and that the functor $\text{pro}_{\mathcal{A}, \mathcal{B}}$ is right adjoint to the forgetful functor. The adjunction morphisms are just the standard ones for extension of scalars. Namely, for an $(\mathcal{A}, K, \mathcal{D})$ -module V , $\Phi_V : V \rightarrow \text{pro}_{\mathcal{A}, \mathcal{B}}(V)$ is given by

$$\Phi_V(v)(a) = \pi_V(a)v, \quad a \in \mathcal{A}, v \in V,$$

while for a $(\mathcal{B}, K, \mathcal{D})$ -module W , $\Psi_W : \text{pro}_{\mathcal{A}, \mathcal{B}}(W) \rightarrow W$ is given by evaluation at 1.

The composition of $\text{pro}_{\mathcal{A}, \mathcal{U}(\mathfrak{k})}$ and Ind_w gives the desired right adjoint to the forgetful functor from $\mathcal{M}(\mathcal{A}, K, \mathcal{D})$ to $\mathcal{M}(\mathcal{D})$. Furthermore, the same is true on the level of homotopic categories, as is easy to check using [P2], §2.4.

It is obvious that Ind_w preserves acyclicity. We want to prove the same for $\text{pro}_{\mathcal{A}, \mathcal{U}(\mathfrak{k})}$. Here we need the assumption that K is reductive, but also an assumption on \mathcal{A} , namely that \mathcal{A} is freely generated over $\mathcal{U}(\mathfrak{k})$ by a (basis of a) K -submodule. This is a stronger assumption than the one in [BL2], where it is only assumed that \mathcal{A} is projective over $\mathcal{U}(\mathfrak{k})$. However, our assumption is satisfied in the most interesting example: $\mathcal{A} = \mathcal{U}(\mathfrak{g})$, where (\mathfrak{g}, K) is a classical Harish-Chandra pair with K reductive, i.e., K acts algebraically on \mathfrak{g} via inner automorphisms, \mathfrak{k} embeds into \mathfrak{g} , and the differential of the K -action is ad composed with the embedding of \mathfrak{k} into \mathfrak{g} ; see [P2], Section 1.1.

2.2. Lemma. *Let (\mathfrak{g}, K) be a Harish-Chandra pair, with K reductive. Then, as a $\mathcal{U}(\mathfrak{k})$ -module for the left multiplication,*

$$\mathcal{U}(\mathfrak{g}) = \mathcal{U}(\mathfrak{k}) \otimes_{\mathbb{C}} \mathcal{P},$$

where \mathcal{P} is a K -invariant subspace of $\mathcal{U}(\mathfrak{g})$.

Proof. Let \mathfrak{p} be a K -invariant complement of \mathfrak{k} in \mathfrak{g} . Let X_1, \dots, X_n be a basis of \mathfrak{k} and let Y_1, \dots, Y_m be a basis of \mathfrak{p} . Then

$$X^i Y^j; \quad i \in \mathbb{Z}_+^n, j \in \mathbb{Z}_+^m$$

is a Poincaré-Birkhoff-Witt basis of $\mathcal{U}(\mathfrak{g})$. Let $\sigma : \mathcal{S}(\mathfrak{p}) \rightarrow \mathcal{U}(\mathfrak{g})$ be the symmetrization map:

$$\sigma(Z_1 \dots Z_k) = \frac{1}{k!} \sum_{s \in S_k} Z_{s(1)} \dots Z_{s(k)}$$

for $Z_1, \dots, Z_k \in \mathfrak{p}$. Then it is easy to check (see [LM], Lemma 2.2) that

$$X^i \sigma(Y^j); \quad i \in \mathbb{Z}_+^n, j \in \mathbb{Z}_+^m$$

is still a basis of $\mathcal{U}(\mathfrak{g})$, and that the span \mathcal{P} of $\{\sigma(Y^j); j \in \mathbb{Z}_+^m\}$ is K -invariant. \square

Now if $\mathcal{A} = \mathcal{U}(\mathfrak{k}) \otimes_{\mathbb{C}} \mathcal{P}$ as a $\mathcal{U}(\mathfrak{k})$ -module for the left multiplication, where \mathcal{P} is a K -invariant subspace of \mathcal{A} , then as a K -module,

$$\text{pro}_{\mathcal{A}, \mathcal{B}}(V) = \text{Hom}_{\mathbb{C}}(\mathcal{P}, V)^{\text{alg}},$$

depends only on the K -module structure of V . Since acyclicity of a complex can be checked on the level of K -modules, it is enough to show that the functor $V \mapsto \text{Hom}_{\mathbb{C}}(\mathcal{P}, V)^{\text{alg}}$ from $\mathcal{M}(K)$ into $\mathcal{M}(K)$ is exact (here $\mathcal{M}(K)$ is the category of algebraic K -modules). This is however true since $\mathcal{M}(K)$ is a semisimple category for reductive K . Hence we have proved

2.3. Proposition. *Assume that K is reductive and that \mathcal{A} is freely generated by a K -submodule as a $\mathcal{U}(\mathfrak{k})$ -module for the left multiplication. Then the forgetful functor from the category of $(\mathcal{A}, K, \mathcal{D})$ -modules to the category of DG modules over \mathcal{D} preserves K -projectives.*

We are ready now to prove the Bernstein-Lunts equivalence. Let $\varepsilon : \mathcal{D} \rightarrow \mathcal{E}$ be a K -equivariant morphism of DG algebras, which is a quasiisomorphism, i.e., induces an isomorphism in cohomology. We also assume that the structural maps $\rho_{\mathcal{E}} : \mathfrak{k} \rightarrow \mathcal{E}$ and $\rho_{\mathcal{D}} : \mathfrak{k} \rightarrow \mathcal{D}$ satisfy $\rho_{\mathcal{E}} = \varepsilon \circ \rho_{\mathcal{D}}$. The main example is the counit map $N(\mathfrak{k}) \rightarrow \mathbb{C}$, where $N(\mathfrak{k})$ is the standard complex of the Lie algebra \mathfrak{k} (see [P2], 2.2.2 and 2.2.4). Then ε induces a forgetful functor $\text{For} : \mathcal{M}(\mathcal{A}, K, \mathcal{E}) \rightarrow \mathcal{M}(\mathcal{A}, K, \mathcal{D})$, which is just the restriction of scalars. This functor has a left adjoint

$$V \longmapsto V \otimes_{\mathcal{D}} \mathcal{E}$$

as is proved in [P2], §2.5. Recall that the \mathcal{A} -action on $V \otimes_{\mathcal{D}} \mathcal{E}$ is the given action on the first factor, the K -action is on both factors, while the \mathcal{E} -action is given by the right multiplication in the second factor, twisted to a left action. Both functors make sense and remain adjoint on the level of homotopic categories.

Clearly, For preserves acyclicity, i.e., preserves quasiisomorphisms, and therefore defines a functor on the level of derived categories. By 1.4, there are enough K-projective $(\mathcal{A}, K, \mathcal{D})$ -modules, so by A1.4.4 and A1.3.3 the functor $V \mapsto V \otimes_{\mathcal{D}} \mathcal{E}$ has a left derived functor, which we denote by $V \mapsto V^L \otimes_{\mathcal{D}} \mathcal{E}$. By A2.3.2, this functor is left adjoint to $\text{For} : D(\mathcal{A}, K, \mathcal{E}) \rightarrow D(\mathcal{A}, K, \mathcal{D})$.

2.4. Theorem. *Assume that K is reductive and that \mathcal{A} is freely generated by a K -submodule as a $\mathcal{U}(\mathfrak{k})$ -module for the left multiplication. Then the functors For and $V \mapsto V^L \otimes_{\mathcal{D}} \mathcal{E}$, induced by $\varepsilon : \mathcal{D} \rightarrow \mathcal{E}$ as above, are mutually inverse equivalences of categories $D(\mathcal{A}, K, \mathcal{E})$ and $D(\mathcal{A}, K, \mathcal{D})$.*

Proof. Let V be an $(\mathcal{A}, K, \mathcal{D})$ -module, and let $P \xrightarrow{\delta} V$ be a K-projective resolution of V . Then $V^L \otimes_{\mathcal{D}} \mathcal{E} = P \otimes_{\mathcal{D}} \mathcal{E}$, and the adjunction morphism $\Phi_V : V \rightarrow \text{For}(V^L \otimes_{\mathcal{D}} \mathcal{E})$ is given by the triple

$$V \xleftarrow[\sim]{\delta} P \xrightarrow{id \otimes 1} \text{For}(P \otimes_{\mathcal{D}} \mathcal{E})$$

(\sim denotes a quasiisomorphism). Namely, by A2.1.2, $\Phi_V = \alpha_{V, V^L \otimes_{\mathcal{D}} \mathcal{E}}(1_{V^L \otimes_{\mathcal{D}} \mathcal{E}})$; however, by the proof of A2.3.2, this can be identified with

$$V \xleftarrow{\delta} P \xrightarrow{\bar{\alpha}_{P, P \otimes_{\mathcal{D}} \mathcal{E}}(1_{P \otimes_{\mathcal{D}} \mathcal{E}})} \text{For}(P \otimes_{\mathcal{D}} \mathcal{E}),$$

and $\bar{\alpha}_{P, P \otimes_{\mathcal{D}} \mathcal{E}}(1_{P \otimes_{\mathcal{D}} \mathcal{E}}) = \bar{\Phi}_P = id \otimes 1$. Here $\bar{\alpha}$ and $\bar{\Phi}$ refer to the adjunction of $- \otimes_{\mathcal{D}} \mathcal{E}$ and For .

So we only need to show that $id \otimes 1 : P \rightarrow P \otimes_{\mathcal{D}} \mathcal{E}$ is a quasiisomorphism of $(\mathcal{A}, K, \mathcal{D})$ -modules. This morphism factors as

$$P \xrightarrow{id \otimes 1} P \otimes_{\mathcal{D}} \mathcal{D} \xrightarrow{id \otimes \varepsilon} P \otimes_{\mathcal{D}} \mathcal{E},$$

since $\varepsilon(1) = 1$. The first of these two morphisms is an isomorphism; its inverse is given by the action map (this is the trivial change of DG algebras, from \mathcal{D} to \mathcal{D}). To show that the second morphism is a quasiisomorphism, we can pass to DG modules over \mathcal{D} (the property of being a quasiisomorphism can be checked on the level of complexes of vector spaces).

By 2.3, P is a K-projective DG module over \mathcal{D} . Therefore it is also a K-flat DG module over \mathcal{D} (see [BL1], 10.12.4.4). In other words, the functor $V \mapsto P \otimes_{\mathcal{D}} V$, $V \in \mathcal{M}(\mathcal{D})$ preserves acyclicity, or equivalently, preserves quasiisomorphisms. Since $\varepsilon : \mathcal{D} \rightarrow \mathcal{E}$ is a quasiisomorphism of DG \mathcal{D} -modules, so is $id \otimes \varepsilon : P \otimes_{\mathcal{D}} \mathcal{D} \rightarrow P \otimes_{\mathcal{D}} \mathcal{E}$.

So the adjunction morphism Φ_V is an isomorphism, for any $V \in D(\mathcal{A}, K, \mathcal{D})$. Consider now the other adjunction morphism, $\Psi_W : \text{For}(W^L \otimes_{\mathcal{D}} \mathcal{E}) \rightarrow W$, for an $(\mathcal{A}, K, \mathcal{E})$ -module

W . The functor For clearly has the following property: a morphism f in $D(\mathcal{A}, K, \mathcal{E})$ is an isomorphism if and only if $\text{For } f$ is an isomorphism in $D(\mathcal{A}, K, \mathcal{D})$ (in other words, For preserves cohomology). Thus, to check that Ψ_W is an isomorphism in the derived category, it is enough to check that $\text{For } \Psi_W$ is an isomorphism. However, by adjunction, $\text{For } \Psi_W \circ \Phi_{\text{For } W} = id_{\text{For } W}$. By the first part of the proof, $\Phi_{\text{For } W}$ is an isomorphism, hence so is $\text{For } \Psi_W$. This finishes the proof. \square

We now specialize to the case mentioned before, namely the counit morphism $\varepsilon : N(\mathfrak{k}) \rightarrow \mathbb{C}$. This is well known to be a quasiisomorphism, and it clearly also satisfies our other assumptions. The category $D(\mathcal{A}, K, N(\mathfrak{k}))$ is the equivariant derived category $D(\mathcal{A}, K)$ of (\mathcal{A}, K) -modules, and $D(\mathcal{A}, K, \mathbb{C})$ is the ordinary derived category $D(\mathcal{M}(\mathcal{A}, K))$ of (\mathcal{A}, K) -modules. The forgetful functor is the functor $Q : D(\mathcal{M}(\mathcal{A}, K)) \rightarrow D(\mathcal{A}, K)$ which assigns to each complex of (\mathcal{A}, K) -modules the same complex with all i_ξ , $\xi \in \mathfrak{k}$ equal to 0. So we get:

2.5. Corollary. *Assume that K is reductive and that \mathcal{A} is freely generated by a K -submodule as a $\mathcal{U}(\mathfrak{k})$ -module for the left multiplication. Then the forgetful functor $Q : D(\mathcal{M}(\mathcal{A}, K)) \rightarrow D(\mathcal{A}, K)$ is an equivalence of categories.*

In this case we can eliminate the assumption of K being reductive (and also weaken the assumptions on \mathcal{A}), in the same way as in [MP], 2.14. Namely, it was proved in [MP], 2.11, that the equivariant derived category $D(\mathcal{A}, K)$ is equivalent to the category $D_{\mathcal{M}(\mathcal{A}, K)}(\mathcal{A}, L)$, where L is a Levi factor of K . Here $D(\mathcal{A}, L)$ is the equivariant derived category of (\mathcal{A}, L) -modules, and $D_{\mathcal{M}(\mathcal{A}, K)}(\mathcal{A}, L)$ is the full triangulated subcategory of equivariant (\mathcal{A}, L) -complexes with cohomology in $\mathcal{M}(\mathcal{A}, K)$. This was stated there for $\mathcal{A} = \mathcal{U}(\mathfrak{g})$, and $\mathcal{A} = \mathcal{U}_\theta$, the quotient of $\mathcal{U}(\mathfrak{g})$ corresponding to an infinitesimal character, but the proof clearly works for any \mathcal{A} . Namely, the inverse of the natural forgetful functor was given by the equivariant Zuckerman functor $R\Gamma_{K, L}^{equi}$, whose definition is independent of \mathcal{A} .

Assuming that \mathcal{A} is freely generated over $\mathcal{U}(l)$ by an L -submodule, we can apply 2.5 to see that $Q_L : D(\mathcal{M}(\mathcal{A}, L)) \rightarrow D(\mathcal{A}, L)$ is an equivalence of categories. The restriction of Q_L is then clearly an equivalence of the category $D_{\mathcal{M}(\mathcal{A}, K)}(\mathcal{M}(\mathcal{A}, L))$ (i.e., the subcategory of $D(\mathcal{M}(\mathcal{A}, L))$ of complexes of (\mathcal{A}, L) -modules with cohomology in $\mathcal{M}(\mathcal{A}, K)$) with $D_{\mathcal{M}(\mathcal{A}, K)}(\mathcal{A}, L)$. Now $D_{\mathcal{M}(\mathcal{A}, K)}(\mathcal{A}, L)$ is equivalent to $D(\mathcal{A}, K)$ as is explained above. Analogously, $D_{\mathcal{M}(\mathcal{A}, K)}(\mathcal{M}(\mathcal{A}, L))$ is equivalent to $D(\mathcal{M}(\mathcal{A}, K))$. This was proved in [MP], 1.12. It was stated there only for $\mathcal{A} = \mathcal{U}(\mathfrak{g})$, but the only assumption needed was that \mathcal{A} is a flat $\mathcal{U}(l)$ -module for the right multiplication. In this case, the proof of the Duflo-Vergne formula ([MP], 1.6) for $R^p\Gamma_{K, L}$ goes through without changes, and so then does the rest of [MP], §1.

Finally, it is obvious that the functor from $D(\mathcal{M}(\mathcal{A}, K))$ to $D(\mathcal{A}, K)$ induced by Q_L via the above equivalences is precisely Q_K . So we get:

2.6. Corollary. *Let (\mathcal{A}, K) be a Harish-Chandra pair and L a Levi factor of K . Assume that \mathcal{A} is freely generated over $\mathcal{U}(l)$ by an L -submodule. Then the functor $Q : D(\mathcal{M}(\mathcal{A}, K)) \rightarrow D(\mathcal{A}, K)$ is an equivalence of categories.*

A. APPENDIX: SOME FACTS FROM HOMOLOGICAL ALGEBRA

The purpose of this appendix is to present some known things from homological algebra needed in this paper, and some also in [P2], which are either unavailable in the literature, or exist but not in the best form for our purposes. Some of the proofs are omitted or only sketched here, but they are written in more detail in [P1]. Most of them should in fact be easy to do, knowing the statements and looking at analogous proofs in the "classical" situation, which is well covered by the literature.

In Section A1 we present a generalization of the well known construction of derived functors between derived categories using adapted subcategories, to the setting of triangulated categories (which are not necessarily homotopic categories of abelian categories), and their localizations (which are not necessarily derived categories of abelian categories). In Section A2 we collect a few facts about adjoint functors which are used over and over, both in this paper, and also in [P2].

A1. Derived Functors in Triangulated Category Setting

In this section, \mathcal{C} will be a fixed triangulated category. For the definition and basic properties of triangulated categories see [Ve], [KS] or [GM]. The main examples are homotopic and derived categories of abelian categories; we will refer to these as the "classical situation". The above books mostly treat this classical situation. Since equivariant derived categories are not a priori derived categories of abelian categories, the results from the classical situation do not apply directly, and hence our need for a more general setting.

A1.1. S-systems and null systems. The following lemma contains several elementary facts that shall be needed below, but they are not explicitly stated in [KS] or [GM] except partly in the exercises. The proof is omitted here, but it is written out in detail in [P1].

A1.1.1. Lemma. *Let \mathcal{C} be a triangulated category. Then:*

- (i) *Suppose $X \rightarrow Y \rightarrow Z \rightarrow T(X)$ is a distinguished triangle in \mathcal{C} . Then $X \rightarrow Y \rightarrow U \rightarrow T(X)$ is a distinguished triangle in \mathcal{C} if and only if U is isomorphic to Z .*
- (ii) *Let $X \xrightarrow{f} Y \rightarrow Z \rightarrow T(X)$ be a distinguished triangle in \mathcal{C} . Then f is an isomorphism if and only if Z is isomorphic to 0.*
- (iii) *Let $X_i \xrightarrow{f_i} Y_i \xrightarrow{g_i} Z_i \xrightarrow{h_i} T(X_i)$, $i = 1, 2$, be two distinguished triangles in \mathcal{C} . Then their direct sum, $X_1 \oplus X_2 \xrightarrow{f} Y_1 \oplus Y_2 \xrightarrow{g} Z_1 \oplus Z_2 \xrightarrow{h} T(X_1 \oplus X_2)$, is also a distinguished triangle. Here $f = f_1 \oplus f_2$, etc.*

Conversely, if the direct sum is distinguished, then both triangles are distinguished.

- (iv) *For any two objects X and Y of \mathcal{C} , the triangle*

$$X \xrightarrow{0} Y \xrightarrow{i_2} T(X) \oplus Y \xrightarrow{p_1} T(X)$$

is distinguished. \square

An *S-system* in \mathcal{C} is a saturated localizing class \mathcal{S} compatible with triangulation. The saturation condition is

$$s \in \mathcal{S} \text{ if and only if } \exists \text{ morphisms } u, v \text{ such that } u \circ s \in \mathcal{S}, s \circ v \in \mathcal{S}$$

or, equivalently,

$$s \in \mathcal{S} \text{ if and only if } Q(s) \text{ is an isomorphism.}$$

Here $Q : \mathcal{C} \rightarrow \mathcal{C}[\mathcal{S}^{-1}]$ is the localization functor (recall that $\mathcal{C}[\mathcal{S}^{-1}]$ is a triangulated category and Q is exact).

This saturation condition is usually omitted. However, there are some technical advantages of assuming it and on the other hand it is not restrictive. Namely, if \mathcal{S} is any localizing class in \mathcal{C} and $Q_{\mathcal{S}} : \mathcal{C} \rightarrow \mathcal{C}[\mathcal{S}^{-1}]$ the natural functor, one can define a saturated localizing class

$$\overline{\mathcal{S}} = \{t \in \text{Hom } \mathcal{C} \mid Q_{\mathcal{S}}(t) \text{ is an isomorphism}\}$$

and $\mathcal{C}[\mathcal{S}^{-1}]$ and $\mathcal{C}[\overline{\mathcal{S}}^{-1}]$ are isomorphic.

A *null system* in \mathcal{C} is a subfamily \mathcal{N} of $\text{Ob}(\mathcal{C})$ satisfying the following conditions:

- (N1) $0 \in \mathcal{N}$;
- (N2) $X \in \mathcal{N}$ if and only if $T(X) \in \mathcal{N}$;
- (N3) If $X \rightarrow Y \rightarrow Z \rightarrow T(X)$ is a distinguished triangle in \mathcal{C} and $X \in \mathcal{N}, Y \in \mathcal{N}$, then $Z \in \mathcal{N}$;
- (N4) If $X \oplus Y \in \mathcal{N}$ then $X, Y \in \mathcal{N}$.

In other words, the full subcategory generated by \mathcal{N} is a triangulated subcategory closed under isomorphisms and containing all direct summands of all its objects. Namely the converse of (N4) holds because of (N2), (N3) and Lemma A1.1.1.(iv), so \mathcal{N} is an additive subcategory. It is a triangulated subcategory by (N2) and (N3). From (N3) and A1.1.1.(i) it follows that \mathcal{N} is closed under isomorphisms.

This definition differs from the one in [KS], which does not require (N4) to hold. As we shall see, (N4) corresponds to the saturation condition, so what we call a null system here could be called a saturated null system.

A1.1.2. Examples.

- (s1) All isomorphisms in \mathcal{C} form the smallest S-system in \mathcal{C} .
- (s2) If $F : \mathcal{C}' \rightarrow \mathcal{C}$ is an exact functor and \mathcal{S} an S-system in \mathcal{C} , then $F^{-1}(\mathcal{S}) = \{f \in \text{Hom } \mathcal{C}' \mid F(f) \in \mathcal{S}\}$ is an S-system in \mathcal{C}' .
- (s3) All quasiisomorphisms in the homotopic category of complexes over an abelian category form an S-system.
- (n1) All objects of \mathcal{C} isomorphic to 0 form the smallest null system in \mathcal{C} , denoted by \mathcal{C}_0 .
- (n2) If $F : \mathcal{C}' \rightarrow \mathcal{C}$ is an exact functor and \mathcal{N} a null system in \mathcal{C} , then $F^{-1}(\mathcal{N}) = \{X \in \mathcal{C}' \mid F(X) \in \mathcal{N}\}$ is a null system in \mathcal{C}' .
- (n3) All acyclic complexes in the homotopic category of complexes over an abelian category form a null system.

A1.1.3. Lemma. *Let \mathcal{N} be a null system in \mathcal{C} . Then for a morphism $X \xrightarrow{f} Y$ the following are equivalent:*

- (i) *There is a distinguished triangle $X \xrightarrow{f} Y \rightarrow Z \rightarrow T(X)$ with $Z \in \mathcal{N}$;*
- (ii) *For any distinguished triangle $X \xrightarrow{f} Y \rightarrow Z \rightarrow T(X)$, Z is in \mathcal{N} .*

Proof. Obvious from Lemma A1.1.1.(i) since \mathcal{N} is closed under isomorphisms. \square

Given \mathcal{N} , let us denote by $\mathcal{S}(\mathcal{N})$ the class of all morphisms of \mathcal{C} satisfying the conditions of A1.1.3.

A1.1.4. Proposition. $\mathcal{S}(\mathcal{N})$ is an S-system.

Proof. It is proved in [KS], Prop.1.6.7., that $\mathcal{S}(\mathcal{N})$ is a localizing class. Compatibility with triangulation follows easily from (N2) and (N3). To show that $\mathcal{S}(\mathcal{N})$ is also saturated, we need the following fact:

A1.1.5. Lemma. Let $Q : \mathcal{C} \rightarrow \mathcal{C}[\mathcal{S}(\mathcal{N})^{-1}]$ be the localization functor. Then an object X is in \mathcal{N} if and only if $Q(X)$ is isomorphic to 0.

Proof. Suppose that $Q(X) \cong 0$. It means that $1_{Q(X)} = 0_{Q(X)}$, i.e., $Q(1_X) = Q(0_X)$. By an elementary property of localization, this implies that there is a morphism $Y \xrightarrow{s} X$ in $\mathcal{S}(\mathcal{N})$ such that $1_X \circ s = 0_X \circ s$, that is $s = 0$. So $Y \xrightarrow{0} X$ is in $\mathcal{S}(\mathcal{N})$. Hence there is a distinguished triangle $Y \xrightarrow{0} X \rightarrow Z \rightarrow T(X)$ with $Z \in \mathcal{N}$. Using Lemma A1.1.1.(iv) and (i), and the fact that \mathcal{N} is closed under isomorphisms, we conclude that $T(Y) \oplus X$ is in \mathcal{N} . However, then X is also in \mathcal{N} by (N4).

Conversely, let X be in \mathcal{N} . Then from the distinguished triangle $X \xrightarrow{1} X \rightarrow 0 \rightarrow T(X)$ we conclude that the morphism $X \rightarrow 0$ is in $\mathcal{S}(\mathcal{N})$, so the corresponding morphism $Q(X) \rightarrow Q(0) = 0$ is an isomorphism. \square

Now we can show that $\mathcal{S}(\mathcal{N})$ is saturated. Suppose $Q(s)$ is an isomorphism. Let $X \xrightarrow{s} Y \rightarrow Z \rightarrow T(X)$ be a distinguished triangle. We have to prove that Z is in \mathcal{N} . However, applying Q to the above triangle we get $Q(Z) \cong 0$ (by A1.1.1.(ii)). So $Z \in \mathcal{N}$ by A1.1.5. This finishes the proof of A1.1.4. \square

The following lemma is an easy consequence of A1.1.1.(ii).

A1.1.6. Lemma. Let \mathcal{S} be an S-system in \mathcal{C} and let Q be the corresponding localization functor. Then the following are equivalent for an object N of \mathcal{C} :

- (i) $Q(N) \cong 0$;
- (ii) For any distinguished triangle $X \xrightarrow{f} Y \rightarrow N \rightarrow T(X)$, f is in \mathcal{S} ;
- (iii) There is a distinguished triangle $X \xrightarrow{f} Y \rightarrow N \rightarrow T(X)$ with $f \in \mathcal{S}$. \square

Given \mathcal{S} , let $\mathcal{N}(\mathcal{S})$ be the full subcategory of all objects satisfying the conditions of A1.1.6. We shall sometimes call these objects \mathcal{S} -acyclic.

A1.1.7. Proposition. $\mathcal{N}(\mathcal{S})$ is a null system.

Proof. Follows from A1.1.2.(n1), (n2). Namely, $\mathcal{N}(\mathcal{S})$ is the inverse under Q of the null system $\mathcal{C}[\mathcal{S}^{-1}]_0$. \square

A1.1.8. Theorem. *Attaching $\mathcal{S}(\mathcal{N})$ to \mathcal{N} and $\mathcal{N}(\mathcal{S})$ to \mathcal{S} gives a one-to-one correspondence between S-systems and null systems in \mathcal{C} .*

Proof. Clearly $\mathcal{N} \subset \mathcal{N}(\mathcal{S}(\mathcal{N}))$ and $\mathcal{S} \subset \mathcal{S}(\mathcal{N}(\mathcal{S}))$. $\mathcal{N}(\mathcal{S}(\mathcal{N})) \subset \mathcal{N}$ is Lemma A1.1.3. So it remains to prove that $\mathcal{S}(\mathcal{N}(\mathcal{S})) \subset \mathcal{S}$. However, $s \in \mathcal{S}(\mathcal{N}(\mathcal{S}))$ implies the existence of a distinguished triangle $X \xrightarrow{s} Y \rightarrow N \rightarrow T(X)$ with $N \in \mathcal{N}(\mathcal{S})$. So $Q_{\mathcal{S}}(N) \cong 0$ and therefore $Q_{\mathcal{S}}(s)$ is an isomorphism by A1.1.1.(ii). Hence $s \in \mathcal{S}$ since \mathcal{S} is saturated. \square

A1.1.9. Remark. Under the correspondence from A1.1.8., the examples from A1.1.2. correspond to each other as follows:

- (1) A1.1.2.(s1) corresponds to A1.1.2.(n1) (by Lemma A1.1.1.(ii).)
- (2) A1.1.2.(s2) corresponds to A1.1.2.(n2), meaning that if \mathcal{N} and \mathcal{S} in \mathcal{C} correspond to each other, then $F^{-1}(\mathcal{N})$ and $F^{-1}(\mathcal{S})$ also correspond to each other.
- (3) A1.1.2.(s3) corresponds to A1.1.2.(n3) .

We also remark that instead of null systems one can consider *thick subcategories* of \mathcal{C} , as it is done in [Ve] and [GM]. It is however easy to prove that this notion is the same as our notion of a null system.

A1.2. Derived functors and acyclic functors. Let \mathcal{C} be a triangulated category, \mathcal{S} an S-system in \mathcal{C} , \mathcal{N} the corresponding null system of \mathcal{S} -acyclic objects and $Q : \mathcal{C} \rightarrow \mathcal{C}[\mathcal{S}^{-1}]$ the localization functor.

Let $(\mathcal{C}', \mathcal{S}', \mathcal{N}')$ be another triple as above, and let $F : \mathcal{C} \rightarrow \mathcal{C}'$ be an exact functor. We call F $(\mathcal{S}, \mathcal{S}')$ -acyclic if $F(\mathcal{S}) \subset F(\mathcal{S}')$, or equivalently $F(\mathcal{N}) \subset F(\mathcal{N}')$. These two properties are equivalent by the description of the correspondence between S-systems and null systems.

As an example, consider two abelian categories \mathcal{A} and \mathcal{A}' . Let \mathcal{S} and \mathcal{S}' be the S-systems of all quasiisomorphisms in $K(\mathcal{A})$ and $K(\mathcal{A}')$, and let $\mathcal{N}, \mathcal{N}'$ be the null systems of acyclic complexes (see A1.1.9.(3)). Then for any functor $F : \mathcal{A} \rightarrow \mathcal{A}'$ which is exact (in the classical sense), $K(F)$ is $(\mathcal{S}, \mathcal{S}')$ -acyclic.

For the rest of this section we assume that \mathcal{N}' is the null system \mathcal{C}'_0 of all objects isomorphic to 0 and \mathcal{S}' the corresponding S-system of all isomorphisms (see A1.1.9.(1)). In this case, F is called just \mathcal{S} -acyclic.

Let $F : \mathcal{C} \rightarrow \mathcal{C}'$ be an arbitrary exact functor. A right derived functor of F is a pair (RF, ε_F) , where

$$RF : \mathcal{C}[\mathcal{S}^{-1}] \rightarrow \mathcal{C}'$$

is an exact functor and

$$\varepsilon_F : F \rightarrow RF \circ Q$$

is a morphism of functors¹, such that the following universal property holds: if (G, ε) is another such pair, then there exists a unique morphism of functors $\eta : RF \rightarrow G$ such that

¹The definition of a functor F between triangulated categories includes a fixed isomorphism from TF into FT , where T is the translation functor. A morphism of functors is assumed to be compatible with these isomorphisms.

the diagram

$$\begin{array}{ccc} F & \xrightarrow{\varepsilon_F} & RF \circ Q \\ =\downarrow & & \eta \circ Q \downarrow \\ F & \xrightarrow{\varepsilon} & G \circ Q \end{array}$$

commutes. If RF exists, it is unique up to isomorphism. This follows from the universal property in a standard way.

Dually, one defines a left derived functor of F : it is a pair (LF, ε_F) , where $LF : \mathcal{C}[S^{-1}] \rightarrow \mathcal{C}'$ is an exact functor and $\varepsilon_F : LF \circ Q \rightarrow F$ is a morphism of functors, such that the following universal property holds: if (G, ε) is another such pair, then there exists a unique morphism of functors $\eta : G \rightarrow LF$ such that $\varepsilon = \varepsilon_F \circ (\eta \circ Q)$.

These definitions in this generality are due to P.Deligne, [De].

A right (or left) derived functor of F does not have to exist. If however F is S -acyclic, then by the universal property of localization there is a unique functor $\bar{F} : \mathcal{C}[S^{-1}] \rightarrow \mathcal{C}'$ such that $F = \bar{F} \circ Q$. Furthermore, \bar{F} is exact, and it is easy to prove that

A1.2.1. Proposition. *If F is S -acyclic, then \bar{F} is a right derived functor of F , if we take $\varepsilon_F = 1 : F \rightarrow \bar{F} \circ Q$ (and also left derived, with $\varepsilon_F = 1 : \bar{F} \circ Q \rightarrow F$). \square*

Since \bar{F} is just the factorization of F through the localized category, it is usually denoted again by F to simplify notation.

A1.2.2. Remark. In the more general situation when we have an arbitrary S-system S' in \mathcal{C}' , we consider the composition $Q_{S'} \circ F$ and we define the right (left) derived functor of F to be the right (left) derived functor of $Q_{S'} \circ F$. In the latter case the above definition applies.

A1.3. Adapted subcategories. If F is not acyclic, one tries to find a triangulated subcategory \mathcal{D} of \mathcal{C} such that $F|_{\mathcal{D}}$ is acyclic. Then F will factor through the localization of \mathcal{D} with respect to the induced S-system $S_{\mathcal{D}}$. In case \mathcal{D} is big enough, every object of $\mathcal{C}[S^{-1}]$ will be isomorphic to an object of $\mathcal{D}[S_{\mathcal{D}}^{-1}]$ and we will be able to construct RF and LF using that. This is the same construction as in the classical situation, but we will review it in some detail to show that it works in our more general situation.

If \mathcal{D} is a triangulated subcategory of \mathcal{C} then \mathcal{D} is a triangulated category with the induced structure: a triangle in \mathcal{D} is distinguished if and only if it is distinguished as a triangle in \mathcal{C} and the translation functor on \mathcal{D} is the restriction of the translation on \mathcal{C} .

A1.3.1. Lemma. *Let \mathcal{D} be a triangulated subcategory of \mathcal{C} , S an S-system in \mathcal{C} and \mathcal{N} the corresponding null system. Then $\mathcal{N}_{\mathcal{D}} = \mathcal{N} \cap \text{Ob } \mathcal{D}$ is a null system in \mathcal{D} and the corresponding S-system $S_{\mathcal{D}}$ is equal to $S \cap \text{Hom } \mathcal{D}$.*

Proof. This is just a special case of A1.1.2.(n2) and A1.1.9.(2) where the functor F is the inclusion from \mathcal{D} into \mathcal{C} . \square

Now we can form the localized category $\mathcal{D}[S_{\mathcal{D}}^{-1}]$ and we get an exact functor $\Psi : \mathcal{D}[S_{\mathcal{D}}^{-1}] \rightarrow \mathcal{C}[S^{-1}]$. Namely, the elements of $S_{\mathcal{D}}$ clearly go to isomorphisms under $\mathcal{D} \rightarrow \mathcal{C} \rightarrow \mathcal{C}[S^{-1}]$.

Let us now further assume that \mathcal{D} satisfies one of the following properties:

- (b_R) For any $X \in \mathcal{C}$ there exists $X \xrightarrow{s} M$ with $s \in \mathcal{S}$ and $M \in \mathcal{D}$
- (b_L) For any $X \in \mathcal{C}$ there exists $M \xrightarrow{s} X$ with $s \in \mathcal{S}$ and $M \in \mathcal{D}$

A1.3.2. Lemma. *Let \mathcal{D} be a triangulated subcategory of \mathcal{C} such that (b_R) or (b_L) holds. Then:*

- (i) Ψ is fully faithful, so $\mathcal{D}[\mathcal{S}_\mathcal{D}^{-1}]$ can be viewed as a full subcategory of $\mathcal{C}[\mathcal{S}^{-1}]$;
- (ii) $\mathcal{D}[\mathcal{S}_\mathcal{D}^{-1}]$ is a triangulated subcategory of $\mathcal{C}[\mathcal{S}^{-1}]$ and its natural triangulated structure (coming from \mathcal{D} and localization) is equal to the induced triangulated structure;
- (iii) Ψ is an equivalence of categories and any of its quasiinverses is exact.

Proof. Done in the classical situation. \square

A1.3.3. Theorem. *Let \mathcal{D} be a triangulated subcategory of \mathcal{C} satisfying (b_R) or (b_L) above. Let $F : \mathcal{C} \rightarrow \mathcal{C}'$ be an exact functor such that*

- (a) *The restriction of F to \mathcal{D} is $\mathcal{S}_\mathcal{D}$ -acyclic.*

Then F has a left derived functor (LF, ε_F) in case (b_L) holds (and F has a right derived functor in case (b_R) holds). Furthermore, for any $M \in \mathcal{D}$, $\varepsilon_F(M)$ is an isomorphism.

Proof. Analogous to the classical situation. \square

We shall say that \mathcal{D} is left (right) adapted to F if it satisfies the conditions (a) and (b_L) ((a) and (b_R)).

A1.4. \mathcal{S} -projective and \mathcal{S} -injective objects.

Let \mathcal{C} be a triangulated category, \mathcal{S} an S-system in \mathcal{C} and \mathcal{N} the corresponding null system of \mathcal{S} -acyclic objects.

A1.4.1. Theorem. *Let P be an object of \mathcal{C} . Then the following conditions are equivalent:*

- (1) *For any $X \in \mathcal{N}$, $\text{Hom}_{\mathcal{C}}(P, X) = 0$;*
- (2) *If $s : X \rightarrow P$ is in \mathcal{S} , then there exists a $t \in \text{Hom}_{\mathcal{C}}(P, X)$ such that $s \circ t = 1_P$ (we will also see that t is unique and in \mathcal{S});*
- (3) *For any $X \in \text{Ob } \mathcal{C}$, the natural homomorphism between abelian groups $\text{Hom}_{\mathcal{C}}(P, X)$ and $\text{Hom}_{\mathcal{C}[\mathcal{S}^{-1}]}(P, X)$ mapping $P \xrightarrow{f} X$ into $P \xleftarrow{1} P \xrightarrow{f} X$ is an isomorphism;*
- (4) *For any diagram*

$$\begin{array}{ccc} & X & \\ & \downarrow s & \\ P & \xrightarrow{f} & Y \end{array}$$

in \mathcal{C} with $s \in \mathcal{S}$, there is a unique $g \in \text{Hom}_{\mathcal{C}}(P, X)$ such that $f = s \circ g$.

Proof. See [Sp], 1.4. and the remark below. \square

A1.4.2. Remark. Suppose there is an exact bifunctor H from $\mathcal{C}^{opp} \times \mathcal{C}$ into the homotopic category of complexes of abelian groups, such that for any two objects X, Y in \mathcal{C}

$$H^i(H(X, Y)) = \text{Hom}_{\mathcal{C}}(X, Y[i]).$$

For example in the classical situation we can take $H = \text{Hom}^\cdot$. Then clearly the condition (1) in A1.4.1. is equivalent to the condition

- (1') The functor $H(P, -)$ maps acyclic objects into acyclic complexes of abelian groups (in the classical sense).

This is the condition that is used in [Sp].

Objects that satisfy the conditions of A1.4.1. are called \mathcal{S} -projective objects. Dually, one defines \mathcal{S} -injective objects. They were first introduced by Verdier in [Ve]; he calls them free on the left, respectively right. Their basic properties and some applications were studied by Spaltenstein in [Sp]. He considered the classical case of the homotopic category $K(\mathcal{A})$ of an abelian category \mathcal{A} , with \mathcal{S} being the class of all quasiisomorphisms and \mathcal{N} the null system of acyclic complexes. To suggest that the definitions are intrinsic to the homotopic category, he called \mathcal{S} -projective objects of $K(\mathcal{A})$ K-projective, and \mathcal{S} -injective objects K-injective. The same terminology is used by Bernstein and Lunts in [BL1] and [BL2] in the case of homotopic category of equivariant complexes. We will follow this terminology in all these cases, but in the present generality the prefix ‘K’ does not make sense.

The importance of \mathcal{S} -projectives and \mathcal{S} -injectives lies in the fact that they can be used as adapted categories to define derived functors, as we are going to explain now.

A1.4.3. Proposition. *The full subcategory \mathcal{P} of \mathcal{C} consisting of \mathcal{S} -projective objects is a null system in \mathcal{C} . The same holds for \mathcal{I} , the full subcategory of \mathcal{S} -injectives. In particular, \mathcal{P} and \mathcal{I} are triangulated subcategories of \mathcal{C} .*

Proof. This follows easily from property (1) in A1.4.1. Namely \mathcal{P} is additive since the functor $\text{Hom}_{\mathcal{C}}(-, X)$ is additive. \mathcal{P} is closed under T and T^{-1} since the null system of \mathcal{S} -acyclic objects is closed under T and T^{-1} , and furthermore $\text{Hom}_{\mathcal{C}}(TP, X) = \text{Hom}_{\mathcal{C}}(P, T^{-1}X)$ and $\text{Hom}_{\mathcal{C}}(T^{-1}P, X) = \text{Hom}_{\mathcal{C}}(P, TX)$.

Since the functor $\text{Hom}_{\mathcal{C}}(-, X)$ is cohomological, whenever $P \rightarrow Q \rightarrow K \rightarrow T(P)$ is a distinguished triangle in \mathcal{C} and P and Q are in \mathcal{P} , then K is also in \mathcal{P} . Finally, using the additivity of $\text{Hom}_{\mathcal{C}}(-, X)$ again, we see that a direct summand of an \mathcal{S} -projective object is \mathcal{S} -projective.

The proof for \mathcal{S} -injectives is analogous. \square

A1.4.4. Theorem. *Suppose that there are enough \mathcal{S} -projectives in \mathcal{C} , i.e., that \mathcal{P} satisfies the condition (b_L) from Section A1.3. Then \mathcal{P} is left adapted to any exact functor F from \mathcal{C} to another triangulated category \mathcal{C}' . An analogous claim is true for \mathcal{S} -injectives.*

Proof. It is enough to check the condition (a) from Theorem A1.3.3. However, if P is \mathcal{S} -projective and \mathcal{S} -acyclic, then $\text{Hom}_{\mathcal{C}}(P, P) = 0$, which implies that P is isomorphic to 0 in \mathcal{C} . Hence $F(P)$ must also be isomorphic to 0. \square

In the classical situation of homotopic categories over abelian categories, one usually uses resolutions by complexes with injective or projective components. Let us clarify how this fits into the framework of the theory studied here. As announced earlier, in this situation we call \mathcal{S} -injective objects K-injective, and \mathcal{S} -projective objects K-projective.

A1.4.5. Proposition. *Let \mathcal{A} be an abelian category. Let P^\cdot be a complex over \mathcal{A} , bounded above, such that all P^j are projective objects of \mathcal{A} . Then P^\cdot is a K-projective object of $K^-(\mathcal{A})$ and $K(\mathcal{A})$.*

Dually, if I^\cdot is a complex over \mathcal{A} , bounded below and having injective components, then I^\cdot is a K-injective object of $K^+(\mathcal{A})$ and $K(\mathcal{A})$.

Proof. To show that P^\cdot is a K-projective object of $K^-(\mathcal{A})$, we have to show that if C^\cdot is an acyclic complex bounded above, then any chain map from P^\cdot to C^\cdot is homotopic to zero. However, this is a well-known fact from classical homological algebra, used to prove that left derived functors are well defined.

To get the claim for $K(\mathcal{A})$, we can just note that since P^\cdot is bounded above, the chain maps from P^\cdot into any complex C^\cdot are actually chain maps into an appropriate truncation of C^\cdot . However, if C^\cdot is acyclic, so is any of its truncations.

The other claim is proved in the same way. \square

An example in the introduction of [Sp] (taken from [Do]) shows that A1.4.5 is not true without the boundedness assumptions. The following fact is well-known:

A1.4.6. Theorem. *Let \mathcal{A} be an abelian category. If \mathcal{A} has enough projectives, then for any X^\cdot in $K^-(\mathcal{A})$ there is a complex P^\cdot with projective components, bounded above by the same degree as X^\cdot , and mapping quasiisomorphically onto X^\cdot . In particular, $K^-(\mathcal{A})$ has enough K-projectives.*

Dually, if \mathcal{A} has enough injectives, any X^\cdot in $K^+(\mathcal{A})$ can be quasiisomorphically embedded into a complex with injective components, bounded below by the same degree as X^\cdot . In particular, $K^+(\mathcal{A})$ has enough K-injectives. \square

It is shown in [Sp] that, under certain conditions, there are also enough K-injectives and K-projectives in $K(\mathcal{A})$. In particular, we need

A1.4.7. Theorem. ([Sp], 3.5) *If an abelian category \mathcal{A} has enough projectives, has direct limits, and the direct limit functor is exact, then $K(\mathcal{A})$ has enough K-projectives. \square*

A2. SOME PROPERTIES OF ADJOINT FUNCTORS

In this section we review some standard and some not so standard facts about adjoint functors. Most of the general theory can be found for example in [ML]. The case of abelian categories is easy and well known. I do not however know any reference for A2.3.2; I learned it from D. Milićić.

A2.1. Definition and general properties. Let \mathcal{A} and \mathcal{B} be two categories. Let F be a functor from \mathcal{A} to \mathcal{B} and let G be a functor from \mathcal{B} to \mathcal{A} . We say that F is left adjoint to G , or that G is right adjoint to F , if the bifunctors $\text{Hom}_{\mathcal{B}}(F(-), -)$ and $\text{Hom}_{\mathcal{A}}(-, G(-))$

from $\mathcal{A}^{opp} \times \mathcal{B}$ into $Sets$ are isomorphic. This condition means that for any two objects X from \mathcal{A} and Y from \mathcal{B} , there is a bijection

$$\alpha = \alpha_{X,Y} : \text{Hom}_{\mathcal{B}}(FX, Y) \xrightarrow{\cong} \text{Hom}_{\mathcal{A}}(X, GY),$$

natural in X and Y . It is easy to show that if F and F' are both left adjoint to a functor G from \mathcal{B} to \mathcal{A} , then F and F' are isomorphic. In other words, we can speak of the (unique) left adjoint of G . Analogously, right adjoints are also unique.

Another easy fact is about the adjoint of a composition. Namely, suppose $F : \mathcal{A} \rightarrow \mathcal{B}$ and $F' : \mathcal{B} \rightarrow \mathcal{C}$ are two functors, with right adjoints G and G' respectively. Then GG' is right adjoint to $F'F$.

A related fact with obvious proof is the following:

A2.1.1. Proposition. *Assume $I : \mathcal{A} \rightarrow \mathcal{B}$ is fully faithful, i.e., \mathcal{A} can be identified with a full subcategory of \mathcal{B} . Let \mathcal{C} be another category, and let $F : \mathcal{C} \rightarrow \mathcal{A}$ be a functor. If H is left adjoint to IF , then HI is left adjoint to F . Similarly, if H is right adjoint to IF , then HI is right adjoint to F .*

Note that HI can be viewed as the restriction of H to \mathcal{A} . \square

A2.1.2. Theorem. *Let F from \mathcal{A} to \mathcal{B} and G from \mathcal{B} to \mathcal{A} be two functors. Then F is left adjoint to G if and only if there are natural transformations*

$$\Phi : Id_{\mathcal{A}} \rightarrow GF, \quad \Psi : FG \rightarrow Id_{\mathcal{B}},$$

such that for any object X of \mathcal{A} , the composition

$$FX \xrightarrow{F(\Phi_X)} FGF(X) \xrightarrow{\Psi_{FX}} FX$$

is the identity morphism, and for any object Y of \mathcal{B} , the composition

$$GY \xrightarrow{\Phi_{GY}} GFG(Y) \xrightarrow{G(\Psi_Y)} GY$$

is the identity morphism.

Proof. If $\alpha : \text{Hom}_{\mathcal{B}}(F(-), -) \xrightarrow{\cong} \text{Hom}_{\mathcal{A}}(-, G(-))$ gives adjunction of F and G , we define Φ and Ψ by $\Phi_X = \alpha_{X,FX}(1_{FX})$ and $\Psi_Y = \alpha_{GY,Y}^{-1}(1_{GY})$.

Conversely, given Φ and Ψ , define α by $\alpha_{X,Y}(\phi) = G(\phi) \circ \Phi_X$. The inverse is given by $\beta_{X,Y}(\psi) = \Psi_Y \circ F(\psi)$.

In filling in the details, naturality plays a key role. \square

The morphisms Φ_X , $X \in \mathcal{A}$ and Ψ_Y , $Y \in \mathcal{B}$ are called adjunction morphisms. The following two statements are easy consequences of A2.1.2; recall that a \mathbb{C} -category is an additive category such that the Hom-groups are not only abelian groups, but have an additional structure of a vector space over \mathbb{C} (compatible with the group structure), and the composition law is bilinear. In the case of \mathbb{C} -categories, additive functors are required to be linear on morphisms.

A2.1.3. Corollary. *Let F from \mathcal{A} to \mathcal{B} be left adjoint to G from \mathcal{B} to \mathcal{A} . If \mathcal{A} and \mathcal{B} are additive categories and if F or G is an additive functor, then the other functor is also additive, and the maps $\alpha_{X,Y}$ are isomorphisms of abelian groups.*

An analogous claim is true for \mathbb{C} -categories. \square

A2.1.4. Proposition. *Let $F : \mathcal{A} \rightarrow \mathcal{B}$ be left adjoint to G and let Φ and Ψ be the corresponding natural transformations. Then:*

- (i) Φ_X is an isomorphism for every $X \in \mathcal{A}$ if and only if F is fully faithful. In that case, G is essentially onto.
- (ii) Ψ_Y is an isomorphism for every $Y \in \mathcal{B}$ if and only if G is fully faithful. In that case, F is essentially onto. \square

A2.1.5. Examples. There are many examples of adjoint functors in various branches of mathematics. Let us mention just a few: the functor assigning to a set X the free group generated by X is left adjoint to the forgetful functor from groups to sets; analogous claims are true in case of algebras, modules, etc. The functor assigning to a Lie algebra its universal enveloping algebra is left adjoint to the forgetful functor from associative algebras into Lie algebras. Limits and colimits also serve as examples of adjunction; see [P1], Section 2.2.

In the following, let us discuss in more detail three classes of examples that are of special interest for us.

(1) Equivalences of categories.

Let $F : \mathcal{A} \rightarrow \mathcal{B}$ be an equivalence of categories. Let $G : \mathcal{B} \rightarrow \mathcal{A}$ be a quasiinverse for F , i.e., GF and FG are isomorphic to the identity functors. Then both F and G are fully faithful and essentially onto; in particular, we have natural isomorphisms

$$\text{Hom}_{\mathcal{B}}(FX, Y) \cong \text{Hom}_{\mathcal{A}}(GFX, GY) \cong \text{Hom}_{\mathcal{A}}(X, GY).$$

So F is left adjoint to G . Analogously, F is right adjoint to G . It is now clear from A2.1.4 that all the corresponding adjunction morphisms are isomorphisms.

Conversely, if F is left adjoint to G and if all adjunction morphisms Φ_X and Ψ_Y are isomorphisms, then F and G are clearly mutually inverse equivalences of categories.

(2) Duality functors.

Let \mathcal{A} be a category and $D : \mathcal{A} \rightarrow \mathcal{A}$ a contravariant functor. We say that D is a duality on \mathcal{A} , if for any X and Y in \mathcal{A} there is an isomorphism

$$\gamma_{X,Y} : \text{Hom}_{\mathcal{A}}(X, DY) \xrightarrow{\cong} \text{Hom}_{\mathcal{A}}(Y, DX),$$

natural in X and Y . Note that D defines two covariant functors, $D' : \mathcal{A}^{opp} \rightarrow \mathcal{A}$ and $D'' : \mathcal{A} \rightarrow \mathcal{A}^{opp}$. The above definition then says that D' is right adjoint to D'' . If we adopt the convention which identifies D with D' , we can say that D is right adjoint to its opposite functor.

The adjunction morphisms corresponding to this adjunction, when interpreted in \mathcal{A} , both give the same morphism $\Phi_X : X \rightarrow DDX$.

Familiar dualities usually satisfy much stronger conditions, like DD being isomorphic to the identity functor, and in case of abelian \mathcal{A} , D being exact. An example of a weaker duality is the standard duality for infinite dimensional vector spaces. It is not hard to show that the natural inclusion $X \rightarrow X^{**}$ gives adjunction as above, but X^{**} is not isomorphic to X . Another example is the duality for finitely generated modules over a commutative ring (or algebra) A :

$$D(M) = \text{Hom}_A(M, A).$$

This is not even an exact functor; however it becomes an equivalence if we pass to the derived category.

(3) Restriction and extension of scalars.

This is another well-known example. Many constructions in this paper and also in [P2] are direct generalizations of this example.

Let $f : A \rightarrow B$ be a morphism of algebras (or just rings). Then there is an obvious ‘restriction of scalars’ functor, or forgetful functor from the category $\mathcal{M}(B)$ of (left) modules over B into the category $\mathcal{M}(A)$: we simply let $a \in A$ act on a B -module M as $f(a)$. This functor For has both adjoints:

The left adjoint is the functor

$$M \longmapsto B \otimes_A M$$

where B is viewed as a right A -module via right multiplication composed with f , and B acts on $B \otimes_A M$ by left multiplication in the first factor. On morphisms, the functor is given by $f \mapsto 1 \otimes f$.

The adjunction morphisms are given as follows: For $M \in \mathcal{M}(A)$, define $\Phi_M : M \rightarrow \text{For}(B \otimes_A M)$ by $\Phi_M(m) = 1 \otimes m$. For $N \in \mathcal{M}(B)$, define $\Psi_N : B \otimes_A M \rightarrow N$ by $\Psi_N(b \otimes n) = bn$, where bn denotes the action of b on n . One now checks that these are well defined morphisms in appropriate categories, and that they really give the required adjunction.

The right adjoint is given by

$$M \longmapsto \text{Hom}_A(B, M)$$

for $M \in \mathcal{M}(A)$. Here B is considered as a left A -module via left multiplication composed with f , and the action of B on $\text{Hom}_A(B, M)$ is given by right multiplication in the first variable.

The adjunction morphisms are defined similarly as for the left adjoint, using the action map and the evaluation at 1. \square

To end this section, let us just mention the fact that if F is left adjoint to G , then F preserves colimits (sums, cokernels, pushouts, direct limits, etc.), while G preserve limits (products, kernels, pullbacks, inverse limits etc.). An important special case of this is the case of functors between abelian categories; then F is right exact (since it preserves cokernels), and G is left exact (since it preserves kernels). These facts are well known; a detailed proof together with the definitions and some more related facts can be found in [P1], §2.2.

A2.2. The case of abelian categories.

In this section, \mathcal{A} and \mathcal{B} are abelian categories, and $F : \mathcal{A} \rightarrow \mathcal{B}$ is left adjoint to $G : \mathcal{B} \rightarrow \mathcal{A}$. As we already mentioned at the end of last section, we have

A2.2.1. Proposition. *F is right exact and G is left exact.*

Recall that an object X of \mathcal{A} is called projective if the functor $\text{Hom}_{\mathcal{A}}(X, -)$ is exact. X is called injective if the functor $\text{Hom}_{\mathcal{A}}(-, X)$ is exact. The following result is very simple, but it is crucial for many constructions in homological algebra.

A2.2.2. Theorem. *If G is exact, F preserves projectives. If F is exact, G preserves injectives.*

Proof. Let $X \in \mathcal{A}$ be projective. Then

$$\text{Hom}_{\mathcal{B}}(FX, -) = \text{Hom}_{\mathcal{A}}(X, G(-))$$

is exact as a composition of two exact functors. Hence FX is projective. The other claim is proved in the same way. \square

A2.2.2 gives a standard way of constructing projectives and injectives in abelian categories. We say that an abelian category \mathcal{A} has enough projectives if every object of \mathcal{A} is a quotient of a projective object. \mathcal{A} has enough injectives if every object of \mathcal{A} embeds into an injective object.

A2.2.3. Theorem. *Suppose that G is exact, that \mathcal{A} has enough projectives, and that the adjunction morphism Ψ_B is an epimorphism for every object B of \mathcal{B} . Then \mathcal{B} has enough projectives.*

Dually, if F is exact, \mathcal{B} has enough injectives, and Φ_A is a monomorphism for every $A \in \mathcal{A}$, then \mathcal{A} has enough injectives.

Proof. Let $B \in \mathcal{B}$. Since \mathcal{A} has enough projectives, there is an epimorphism $P \rightarrow GB$ in \mathcal{A} , with P projective. Applying F , we get an epimorphism $FP \rightarrow FGB$, since F is right exact by A2.2.1. Composing this with $\Psi_B : FGB \rightarrow B$ we get an epimorphism $FP \rightarrow B$. However, FP is projective by A2.2.2. Hence \mathcal{B} has enough projectives. The second claim is proved in the same way. \square

A2.3. The case of triangulated categories.

Let us now go back to the situation of Section A1. Let \mathcal{C} and \mathcal{D} be two triangulated categories with null systems \mathcal{N} and \mathcal{M} and corresponding S-systems \mathcal{S} and \mathcal{T} . Let us denote by $\mathcal{C}_{\mathcal{S}}$ and $\mathcal{D}_{\mathcal{T}}$ the corresponding localizations.

Let $F : \mathcal{C} \rightarrow \mathcal{D}$ and $G : \mathcal{D} \rightarrow \mathcal{C}$ be exact functors. Assume that F is left adjoint to G .

A2.3.1. Theorem. *If G is $(\mathcal{T}, \mathcal{S})$ -acyclic (i.e., sends \mathcal{M} into \mathcal{N}), then F maps \mathcal{S} -projective objects of \mathcal{C} into \mathcal{T} -projective objects of \mathcal{D} . Dually, if F is $(\mathcal{S}, \mathcal{T})$ -acyclic then G maps \mathcal{T} -injectives into \mathcal{S} -injectives.*

Proof. Let $P \in \mathcal{C}$ be \mathcal{S} -projective and let $X \in \mathcal{M}$. Then

$$\text{Hom}_{\mathcal{D}}(F(P), X) = \text{Hom}_{\mathcal{C}}(P, G(X)) = 0$$

since $G(X) \in \mathcal{N}$. So $F(P)$ is \mathcal{T} -projective. The proof of the dual statement is analogous. \square

Next, we want to show that if F and G have derived functors LF and RG , then these derived functors are still adjoint. In fact, to prove this we need to assume that LF and RG can be calculated using adapted subcategories. Of course, this will be true in all practical situations. The proof presented here is due to D. Miličić.

A2.3.2. Theorem. *Let $F : \mathcal{C} \rightarrow \mathcal{D}$ be left adjoint to $G : \mathcal{D} \rightarrow \mathcal{C}$. Let $\mathcal{L} \subset \mathcal{C}$ be a subcategory left adapted to F and let $\mathcal{R} \subset \mathcal{D}$ be a subcategory right adapted to G . Let $LF : \mathcal{C}_{\mathcal{S}} \rightarrow \mathcal{D}_{\mathcal{T}}$ be the left derived functor of F , and let $RG : \mathcal{D}_{\mathcal{T}} \rightarrow \mathcal{C}_{\mathcal{S}}$ be the right derived functor of G ; these functors exist under the above assumptions. Then LF is left adjoint to RG .*

Proof. Let $X \in \mathcal{C}_{\mathcal{S}}$ and $Y \in \mathcal{D}_{\mathcal{T}}$. We have to prove that

$$\text{Hom}_{\mathcal{D}_{\mathcal{T}}}(LF(X), Y) \cong \text{Hom}_{\mathcal{C}_{\mathcal{S}}}(X, RG(Y)),$$

naturally in X and Y . By our assumptions, X is naturally isomorphic in $\mathcal{C}_{\mathcal{S}}$ to an object A of \mathcal{L} , while Y is naturally isomorphic in $\mathcal{D}_{\mathcal{T}}$ to an object B of \mathcal{R} . Also, $LF(X) = F(A)$ and $RG(Y) = G(B)$. Therefore it is enough to show that

$$\text{Hom}_{\mathcal{D}_{\mathcal{T}}}(F(A), B) \cong \text{Hom}_{\mathcal{C}_{\mathcal{S}}}(A, G(B)),$$

naturally in A and B .

Let $\phi : F(A) \rightarrow B$ be a morphism in $\mathcal{D}_{\mathcal{T}}$. We can represent ϕ by a triple

$$F(A) \xrightarrow{f} C \xleftarrow{s} B$$

where f and s are morphisms in \mathcal{D} , and $s \in \mathcal{T}$. We can assume $C \in \mathcal{R}$, passing to an equivalent triple if necessary.

Let $\gamma = \gamma_{A,C}$ be the map from $\text{Hom}_{\mathcal{D}}(F(A), C)$ into $\text{Hom}_{\mathcal{C}}(A, G(C))$ defined by the adjunction of F and G . So γ is natural in A and C . Since B and C are in \mathcal{R} , $G(s)$ is in \mathcal{S} . Therefore the triple

$$A \xrightarrow{\gamma(f)} G(C) \xleftarrow{G(s)} G(B)$$

represents a morphism in $\mathcal{C}_{\mathcal{S}}$ from A to $G(B)$. We denote this morphism by $\alpha(\phi)$. One shows $\alpha(\phi)$ is well-defined, i.e., that it depends only on the morphism ϕ and not on the choice of f and s . Furthermore, $\phi \mapsto \alpha(\phi)$ is natural in A and B .

Finally, in order to prove that $\alpha : \text{Hom}_{\mathcal{D}_{\mathcal{T}}}(F(A), B) \rightarrow \text{Hom}_{\mathcal{C}_{\mathcal{S}}}(A, G(B))$ is a bijection, we construct its inverse. Let $\psi : A \rightarrow G(B)$ be a morphism in $\mathcal{C}_{\mathcal{S}}$. We can represent it by

$$A \xleftarrow{s} C \xrightarrow{f} G(B)$$

with $s \in \mathcal{S}$. As before, we can assume $C \in \mathcal{L}$. We define $\beta(\psi)$ to be the class of the triple

$$F(A) \xleftarrow{F(s)} F(C) \xrightarrow{\gamma^{-1}(f)} B.$$

One now shows that β is indeed inverse to α . \square

We will mostly use the above theorem in special cases, when each of the functors F and G is either acyclic (so we can take the whole category as an adapted category), or its derived functor can be calculated using \mathcal{S} -projectives, respectively \mathcal{T} -injectives. In those cases one could produce slightly simpler proofs for the theorem, but this does not seem to compensate for the loss of generality.

A2.3.3. Remark. Let us show how to apply A2.3.2 to the classical situation. Let $F : \mathcal{A} \rightarrow \mathcal{B}$ be additive and left adjoint to $G : \mathcal{B} \rightarrow \mathcal{A}$, where \mathcal{A} and \mathcal{B} are abelian categories. Then it is a standard (and easy) fact that F and G define exact functors on the level of homotopic categories $K(\mathcal{A})$ and $K(\mathcal{B})$. It is immediate that F is still left adjoint to G on the homotopic level. However, to define LF and RG , one usually has to restrict F to $K^-(\mathcal{A})$ and G to $K^+(\mathcal{B})$ to get the existence of adapted subcategories. In that case A2.3.2 does not even make sense. If however one of the functors, say F , is cohomologically bounded, then LF can be extended to $D(\mathcal{A})$ and then restricted to $D^+(\mathcal{A})$. Inspecting this construction one sees that $LF : D^+(\mathcal{A}) \rightarrow D^+(\mathcal{B})$ is still calculated using a left adapted subcategory. Therefore A2.3.2 can be applied in this situation.

Another possibility is that LF and RG can be defined on full derived categories because of existence of enough K-projectives in $K(\mathcal{A})$ and K-injectives in $K(\mathcal{B})$. Then A2.3.2 applies without any assumptions on F or G .

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